Forward Marching Procedure for Separated Boundary-Layer Flows

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KLINEBERG and Steger[1] and Carter[2] have recently presented finite-difference techniques for solving the laminar, incompressible boundary-layer equations for separated flow. In the reversed flow region, the difference scheme for the streamwise convection term is switched from a backward to a forward difference to account for the change in flow direction. Since the calculation proceeds from the upstream boundary, the use of a forward difference requires the solution at the next downstream station from the one being computed, and hence repeated streamwise iterations are required from separation through reattachement to obtain a converged solution. In many cases this global iteration procedure is unnecessary since the velocity in the reversed flow region is small, typically 5% or less of the freestream velocity. Consequently, neglect of the streamwise convection in the reversed flow region should have only a slight influence on the resulting solution. For example, Reyhner and Flügge-Lotz[3] demonstrated that by setting the convection term uu_x in the x momentum equation equal to zero for u less than zero, a stable finite-difference solution of the boundary-layer equations could be obtained with the usual forward-marching procedure for a separated flow. Not only does this approximation eliminate the well-known instability encountered in solving the boundary-layer equations in a direction opposite to that of the local flow, but it also results in a substantial reduction in computer time and storage as compared to that required for a global iteration procedure.

In the present Note a similar approximation to that of Reyhner and Flügge-Lotz[3] is made by neglecting the streamwise convection of vorticity in the reversed flow region. This approximation is incorporated into the inverse boundary-layer procedure (displacement thickness prescribed) developed previously by Carter.[2] The resulting forward-marching procedure is shown to be a rapid and accurate technique for solving separated flows of limited extent. The equations are solved by the Crank-Nicolson scheme in which column iteration is used at each streamwise station since the finite-difference equations are nonlinear. Instabilities which were encountered in these column iterations were eliminated by introducing timelike terms in the finite-difference equations to provide both unconditional diagonal dominance and a column iterative scheme, found to be stable using the von Neumann stability analysis. This technique is general, and should be applicable to other implicit finite-difference schemes such as the ADI solution procedure for solving the Navier-Stokes equations. Diagonal dominance, as discussed by Keller[4] and in the recent paper by Hirsh and Rudy,[5] is a sufficient condition which insures that error growth will not occur in the solution of the tridiagonal equations with the Thomas algorithm. In some cases Reyhner and Flügge-Lotz,[3] and later Werle et al.[6] in a similar investigation, encountered an unexplained instability in the reverse flow region which they eliminated by introducing a positive artificial convection term. Since this term increases the magnitude of the diagonal coefficients in the tridiagonal system of equations, it is possible that the instability in their solutions was caused by large errors in the Thomas algorithm due to lack of diagonal dominance.

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Problem Formulation

The boundary-layer form of the vorticity transport and stream function equations, respectively, for laminar, incompressible flow are given by

$$u\delta^{*2}(\partial\omega/\partial\xi) - (\delta^*\partial/\partial\xi)$$
$$\times [\tilde{\psi} + (\eta - 1)\ (u\delta^*)](\partial\omega/\partial\eta) = \partial^2\omega/\partial\eta^2 \qquad (1)$$

$$\partial\tilde{\psi}/\partial\eta = \delta^{*2}(1-\eta)\,\omega \qquad (2)$$

where the independent variables are $\xi = x$ and $\eta = y/\delta^*$, and δ^* is the displacement thickness. The transformed stream function $\tilde{\psi}$ is related to the usual stream function ψ by $\tilde{\psi} = \psi - u\delta^*(\eta - 1)$ where u is the x components of velocity. The boundary conditions are given by

$$u(\xi,0) = \tilde{\psi}(\xi,0) = 0 \qquad (3)$$

$$\omega(\xi,\eta) \text{ and } \tilde{\psi}(\xi,\eta) \to 0 \text{ as } \eta \to \infty \qquad (4)$$

Once the vorticity is deduced from Eq. (1) then u is given by

$$u(\xi,\eta) = \delta^* \int_0^\eta \omega(\xi,\eta_1)d\eta_1 \qquad (5)$$

For a given $\delta^*(\xi)$, the use of Eqs. (1-5) results in a nonsingular solution at separation as was originally shown by Catherall and Mangler.[7]

The vorticity equation is solved with the Crank-Nicolson finite-difference scheme which may be written as follows, with $\xi = m\Delta\xi, \eta = n\Delta\eta$, and q denoting the column iteration level:

$$A_n^q \omega_{m,\ n-1}^{q+1} + B_n^q \omega_{m,\ n}^{q+1} + C_n^q \omega_{m,\ n+1}^{q+1} = D_n^q \qquad (6)$$

where

$$A_n = -(C_\eta + 1)/2 \quad B_n = 1 + 2C_\xi \quad C_n = (C_\eta - 1)/2$$

$$D_n = -A_n\omega_{m-1,n} + (2C_\xi - 1)\,\omega_{m-1,n} - C_n\omega_{m-1,n+1}$$

$$C_\xi = (\Delta\eta^2/2\Delta\xi)\ (u\delta^{*2})_{m-1/2,n}$$

$$C_\eta = -(\Delta\eta/2)\,\delta^*_{m-1/2}(\partial/\partial\xi)[\tilde{\psi}$$
$$+ (\eta - 1)\ (u\delta^*)]\,|_{m-1/2,n} \qquad (7)$$

In the reverse flow region the streamwise convection of vorticity is neglected, that is, for $u<0$ $u\delta^{*2}\ \partial\omega/\partial\xi = 0$ which gives $C_\xi = 0$.

Repeated application of Eq. (6), from the wall to the outer boundary results in a tridiagonal system of linear equations for the vorticity. These equations are easily solved by the Thomas algorithm which can be written as

$$\omega_{m,\ n}^{q+1} = D'_n + C'_n\ \omega_{m,\ n-1}^{q+1} \qquad (8)$$

$$D'_n = (D_n - C_n D'_{n+1})/(2B_n + C_n C'_{n+1})$$

$$C'_n = -A_n/(B_n + C_n C'_{n+1}) \qquad (9)$$

The quantities D'_n and C'_n are computed, beginning at the outer boundary, where the boundary condition $\omega(\xi,\infty) = 0$ is imposed and proceeding to the wall. Equation (8) is then used to deduce $\omega_{m,\ n}^{q+1}$ once the value at the wall $\omega_{m,\ 1}^{q+1}$ is known. The wall vorticity is found by simultaneously solving for the stream function from Eq. (2) across the boundary layer and imposing the boundary condition given in Eq. (3). Details of this procedure are presented by Carter.[2] After the back substitution in Eq. (8) is completed, the coefficients in Eq. (6) are updated and the process continued until convergence is obtained. Convergence is assumed when the maximum change in all of the dependent variables between two successive column interations is less than 10^{-5}.

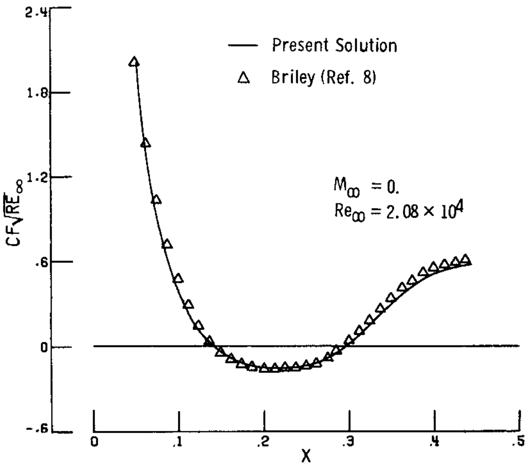

Fig. 1 Comparison of skin friction.

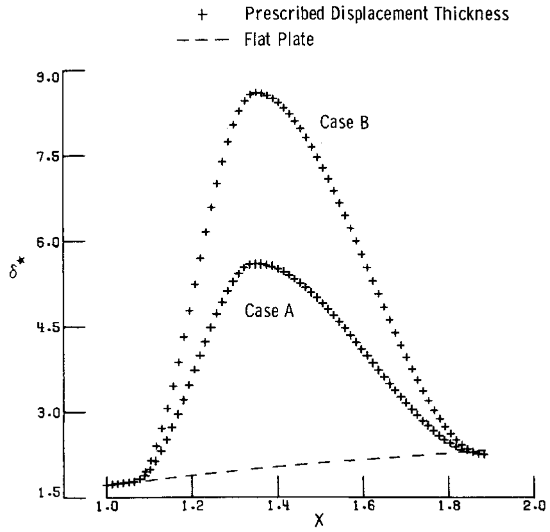

Fig. 2 Prescribed displacement thickness distributions.

Keller[4] presents an inductive proof that if a tridiagonal system of equations are diagonally dominant, that is, $|B_n| \geq |A_n| + |C_n|$ then $|C'_n| \leq 1$, thereby insuring no error growth in applying Eq. (8). Unconditional diagonal dominance can be established by introducing a timelike term in Eq. (6) which gives

$$A^q_n \omega^{q+1}_{m,\,n-1} + (B_n+\alpha)^q \omega^{q+1}_{m,\,n} + C^q_n \omega^{q+1}_{m,\,n+1} = D^q_n + \alpha \omega^q_{m,n} \quad (10)$$

where α is determined so that

$$|B_n + \alpha| \geq |A_n| + |C_n| \quad (11)$$

Examination of Eq. (7) shows if $\alpha = |C_\eta|$, then Eq. (11) is satisfied even if $C_\xi = 0$, which occurs in the reverse flow region. This modification of the finite-difference equations to obtain unconditional diagonal dominance results in a column iterative procedure which may be approximately analyzed by the von Neumann stability analysis. Substituion of a single Fourier component of arbitrary wave number k, $\omega^q_n = \omega^q_0 e^{ik\eta}$ into Eq. (10), where ω_0 is the amplitude, results in the amplification factor

$$|g| = \left| \frac{\omega_0{}^{q+1}}{\omega_0{}^q} \right| = \frac{|\alpha|}{|1 - \cos\phi + 2C_\xi + \alpha + iC_\eta \sin\phi|} \quad (12)$$

where $\phi = k\Delta\eta$. The derivation of Eq. (12) is simplified by assuming D^q_n is a constant, since it depends on A_n, C_n, etc., which are assumed constant, and therefore it is neglected in the stability analysis. Examination of Eq. (12) shows that the column iterative scheme is unconditionally stable since $|g| \leq 1$ provided $C_\xi \geq 0$ and $\alpha \geq 0$, both of which are always satisfied.

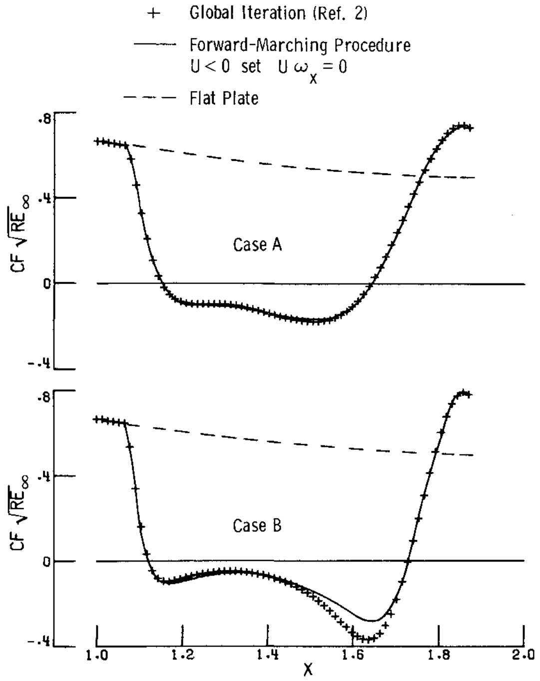

Fig. 3 Comparison of skin-friction distributions for two computational schemes.

Results and Discussion

Figure 1 shows the excellent agreement between the skin friction obtained with the present technique for solving the boundary-layer equations and that found by Briley[8] in solving the Navier-Stokes equations for a flow involving separation and reattachment. The present solution is obtained by specifying the displacement thickness deduced by Briley. The total computer time required in the present calculation for 5700 grid points was about 1 min on the CDC-6600 computer as compared to 45 min used by Briley for 1050 grid points on the UNIVAC 1108. In the present calculation, an average of 10 column iterations were required to obtain convergence.

The displacement thickness distributions shown in Fig. 2 were specified both for the approximate technique described in this Note and for the global iteration procedure presented by Carter,[2] which properly accounts for the reversed flow direction. Comparisons of the resulting skin-friction distributions are shown in Fig. 3. The agreement is quite good in case A where the maximum reverse flow velocity is -0.05 U_∞; in case B where it is about -0.10 U_∞, the accuracy of the approximate solution is slightly less. Cases A and B required about 1 min each on the CDC-6600 using the forward-marching scheme in contrast with the 5-10 min, depending on the initial conditions, required in the global iterative calculation.

Klineberg and Steger[1] converted their global iterative procedure to a forward-marching technique by including the Reyhner and Flügge-Lotz approximation and by using a backward difference on u_x in the continuity equation. However, their calculations were unstable except when a coarse grid was used with a first-order scheme in the stream direction even though the magnitude of the reverse flow velocity was less than 0.02 U_∞. In contrast, in the present calculations, which are for more severe separation cases, no instabilities were encountered by neglecting the streamwise convection of vorticity in the reversed flow region, even with further grid refinements. Since both the reversed flow approximations and the problem formulations are different in these two studies, it is difficult to assess precisely why the present results are stable and those of Klineberg and Steger are not.

In conclusion it is deduced that the present forward-marching technique is quite accurate, provided the magnitude of the reverse flow velocity is less than 0.10 U_∞. In addition, a modification to the Thomas algorithm has been introduced which gives unconditional diagonal dominance and results in a column iterative scheme which has linear stability, at least in the present calculations.

References

[1]Klineberg, J. M., and Steger, J. L., "On Laminar Boundary—Layer Separation," AIAA Paper 74-94, Washington, D.C., 1974.

[2]Carter, J. E., "Solutions for Laminar Boundary Layers with Separation and Reattachment," AIAA Paper 74-583, Palo Alto, Calif., 1974.

[3]Reyhner, T. A. and Flügge-Lotz, I., "The Interaction of a Shock Wave With a Laminar Boundary Layer," *International Journal of Non-Linear Mechanics*, Vol. 3, June 1968, pp. 173-199.

[4]Keller, H. B., *Numerical Methods for Two-Point Boundary-Value Problems*, Chap. 3, Ginn-Blaisdell Pub. Co., Waltham, Mass., 1968, pp. 72-105.

[5]Hirsh, R. S. and Rudy, D. H., "The Role of Diagonal Dominance and Cell Reynolds Number in Implicity Difference Methods for Fluid Mechanics Problems," *Journal of Computational Physics*, Vol. 16 No. 3, 1974, pp. 304-310.

[6]Werle, M. J., Polak, A., and Bertke, S. D., "Supersonic Boundary-Layer Separation and Reattachment—Finite Difference Solutions," Rept. AFL 72-12-1, Jan. 1973, Dept. of Aerospace Engineering, University of Cincinnati, Cincinnati, Ohio.

[7]Catherall, D. and Mangler, K. W., "The Integration of the Two-Dimensional Laminar Boundary-Layer Equations Past the Point of Vanishing Skin Friction," *Journal of Fluid Mechanics*, Vol. 26, part 1, 1966, pp. 163-182.

[8]Briley, W. R., "A Numerical Study of Laminar Separation Bubbles Using the Navier-Stokes Equations," *Journal of Fluid Mechanics*, Vol. 47, part 4, 1971, pp. 713-736.

Bending of Cylindrically Anisotropic Sector Plates

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Introduction

IT is the purpose of this Note to develop deflection and bending stress equations for a cylindrically anisotropic sector which is clamped along its radial edges and may have any boundary conditions along its circular edges. Results are desired for a variety of combinations of boundary conditions, including the case where the inner radius is clamped and the outer radius has a uniform shearing force along its edge. This is the case of a disk-type rotary regenerator matrix subject to a pressure drop and peripheral rubbing seal load.[1-3]

A thin plate of the form shown in Fig. 1 has an inner radius a and an outer radius b; the sector angle is θ. Coordinate r is measured from point O, and coordinate θ is measured from the center radial line of the sector with maximum and minimum values at $\theta = \pm\alpha/2$. The applied force is a uniformly distributed load acting normal to the surface of the plate and having a value of q psi. The boundary conditions are clamped on the circular edges and general along the radial lines.

The deflection is described in terms of two separate functions of the two variables, r and θ. One of the functions is assumed such that the boundary conditions will be satisfied, and the theorem of minimum potential energy[4,5] is used to obtain a differential equation in terms of the other variable alone. Solving this equation provides the two separate functions which, when combined, give an expression for the deflection.

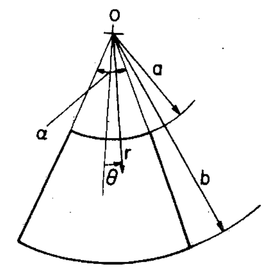

Fig. 1 Sector plate.

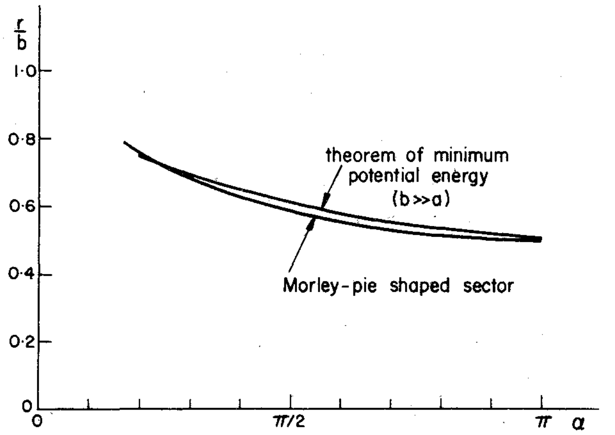

Fig. 2 Location of maximum plate deflection vs sector angle. (Completely clamped, uniformly loaded, isotropic sector, $\nu = 0.3$.)

Some preliminary discussion on the choice of functions is necessary here. The expression for the deflection is:

$$w(r,\theta) = f(r)g(\theta) \tag{1}$$

where $f(r)$ and $g(\theta)$ are functions of the single variables r and θ, respectively. The functions $f(r)$ and $g(\theta)$ must satisfy all the boundary conditions after being multiplied. In general, $f(r)$ must also be a function of the sector angle, α, since it would not be expected that the deflection curve along a line of constant θ would be similar for all sector angles. In fact, Morley[5] has shown that the location of the maximum deflection along the center radial line of an isotropic clamped pie-shaped sector will vary with the sector angle (see Fig. 2). Ben-Amoz assumes a function $f(r)$ for a clamped isotropic pie-shaped sector and includes what appears to be an empirical function of the sector angle in this function. This assumption gives good results until the sector angle becomes small. As the sector angle approaches zero, the maximum deflection approaches the outer radius. This clearly cannot occur for a clamped sector. In the case of the anisotropic plate, the problem becomes more complex since the function $f(r)$ should depend on the anisotropy of the plate as well as the sector angle.

To avoid the above problems, the authors have assumed a function for $g(\theta)$ instead of $f(r)$. There are several advantages to this. The deflections and stresses will be symmetric about the center radial line; thus a simple symmetric function of θ may be chosen without any concern for the sector angle or anisotropy. In addition, when the function $f(r)$ is obtained using the theorem of minimum potential energy, it will depend on all the system parameters that it should depend on, including the sector angle and the plate anisotropy. The results are compared to Morley's data for the case of the isotropic plate and good correlation is obtained.

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